

Exotic Symplectic Manifolds from Lefschetz Fibrations

by

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Bachelor of Arts, University of California, Berkeley, June 2004

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Abstract

In this thesis I construct, in all odd complex dimensions, pairs of Liouville domains W_0 and W_1 which are diffeomorphic to the cotangent bundle of the sphere with one extra subcritical handle, but are not symplectomorphic. While W_0 is symplectically very similar to the cotangent bundle itself, W_1 is more unusual. I use Seidel's exact triangles for Floer cohomology to show that the wrapped Fukaya category of W_1 is trivial. As a corollary we obtain that W_1 contains no compact exact Lagrangian submanifolds.

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Chapter 1

Introduction.

This thesis is concerned with the topology of an important class of symplectic manifolds known as Liouville domains or, from the perspective of complex geometry, Stein domains. A key problem in this area is to understand the complexity of such manifolds; this requires new methods to construct examples, and new invariants to distinguish them. The main tools used for these purposes are Lefschetz fibrations and Floer cohomology.

Donaldson introduced symplectic Lefschetz pencils in [5] and demonstrated that every symplectic manifold admits such a structure. Shortly afterwards Lefschetz theory emerged as a significant tool in symplectic topology, as manifest in the work of Auroux, Akbulut-Ozbagci, Gompf, Smith, Seidel and others. An important feature of Lefschetz fibrations is in encoding the topology of the total space in terms of the fiber and a collection of submanifolds in it, the so called vanishing cycles. This dimensional reduction is particularly striking in dimension four, where the fibers are surfaces. In what may be called an opposite direction, it allows one to construct examples of symplectic manifolds from collections of vanishing cycles in the fiber. In particular, modifying a given collection one gets various families of total spaces. This thesis is concerned with understanding symplectic invariants of the resulting manifolds, in particular their wrapped Floer cohomology and wrapped Fukaya category. We prove the following theorem.

Theorem 1.0.1. *There exist Liouville domains W_0 and W_1 where W_i is obtained by attaching an n -handle to T^*S^{n+1} (n even, $n \geq 2$), with the following properties:*

- W_0 and W_1 are diffeomorphic
- W_0 and W_1 carry Lefschetz fibrations over the disc, such that the wrapped Floer cohomologies of the Lefschetz thimbles are non-zero for W_0 and zero for W_1 .

This has the following

Corollary 1.0.2. *W_1 does not contain any compact exact Lagrangian submanifolds (whereas W_0 contains a Lagrangian S^{n+1}).*

In particular W_0 and W_1 are not exact deformation equivalent; W_1 is the “exotic” version of W_0 . We conjecture that such doppelgängers with vanishing wrapped Fukaya categories exist for wide classes of Liouville domains.

The rest of this thesis is structured as follows. In section 2 we introduce Liouville domains and summarize their basic properties. Section 3 is concerned with exact Lefschetz fibrations for Liouville domains. We discuss thimbles, vanishing cycles, and matching cycles. In section 4 we proceed to construct our main objects of study - the Liouville domains W_0 and W_1 as total spaces of bifibrations. Section 5 reviews the basics of wrapped Floer cohomology and wrapped Fukaya categories as relevant to Lefschetz fibrations. Section 6 recalls Seidel’s exact triangles in Floer cohomology and applies them to W_0 and W_1 , proving the main non-symplectomorphism result. In the final section we put this thesis into a general framework of computations of Floer-theoretic invariants of Liouville domains and discuss extensions of the present work to a more general setting.

Chapter 2

Liouville domains and exact symplectic manifolds with corners.

We study the symplectic topology of Liouville domains. The introduction below closely follows [15].

Definition 2.0.3. A Liouville domain is a compact manifold with boundary M^{2n} , together with a one-form θ which has the following two properties. Firstly, $\omega = d\theta$ should be symplectic. Secondly, the vector field Z defined by $i_Z\omega = \theta$ should point strictly outwards along ∂M .

Example 2.0.4. A Stein manifold U with complex structure J admits an exhausting function $h : U \rightarrow \mathbb{R}$ which is strictly plurisubharmonic, meaning that $-dd_c h = -d(dh \cdot J)$ is a Kähler form. Then, if C is a regular value of h , the sublevel set $M = h^{-1}((-\infty; C])$ is a Liouville domain with $\theta = d_c h$ and the Liouville vector field Z is the gradient of h with respect to the Kähler metric.

Note that $\alpha = \theta|_{\partial M}$ is a contact form on ∂M , and the negative time flow of Z defines a canonical collar neighborhood $\kappa : (-\infty, 0] \times \partial M \rightarrow M$, with $\kappa^*\theta = e^r\alpha$, $\kappa^*Z = \partial_r$. This collar is modeled on the negative part of symplectization of $(\partial M, \alpha)$ and allows us to complete M by attaching an infinite cone corresponding to the positive half

$$\hat{M} = M \cup (\partial M \times [0, \infty)); \hat{\theta}|([0, \infty) \times M) = e^r\alpha; \hat{Z}|([0, \infty) \times M) = \partial_r$$

A Liouville isomorphism between domains M_0 and M_1 is a diffeomorphism $\phi : \hat{M}_0 \rightarrow \hat{M}_1$ satisfying $\phi^*\theta_1 = \theta_0 + d(\text{some compactly supported function})$.

Note that any such ϕ is a symplectomorphism and is compatible with the Liouville flow at infinity. This means that on a piece of the cone $[\rho, \infty) \times \partial M_0 \subset \hat{M}_0$ for some $\rho > 0$, it has the form $\phi(r, y) = (r - f(y), \psi(y))$, where $\psi : \partial M_0 \rightarrow \partial M_1$ is a contact isomorphism, satisfying $\psi^*\alpha_1 = e^f\alpha_0$ for some function f . While the contact structure at the boundary is preserved under Liouville isomorphism, the contact one form is not, and in fact can be changed arbitrarily.

Example 2.0.5. Let N be a manifold and T^*N its cotangent bundle with the standard symplectic form $\omega = d\lambda = \Sigma dp \wedge dq$. Then the vector field $Z = \Sigma p \frac{\partial}{\partial p}$ generates the Liouville flow of “radial rescaling”. Any choice of metric on N makes the corresponding unit disc bundle into a Liouville domain. All such domains are Liouville isomorphic with corresponding completions symplectomorphic to T^*N itself.

A version of Moser’s Lemma, which says that deformation equivalence implies Liouville isomorphism, holds in this context.

Lemma 2.0.6. *Let $(\theta_t)_{0 \leq t \leq 1}$ be a family of Liouville structures on M . Then all the (M, θ_t) are mutually Liouville isomorphic.*

Example 2.0.7. If in the Example 2.0.4 the critical point set of $h : U \rightarrow \mathbb{R}$ is compact, then taking C to be bigger than the largest critical value, we get a Liouville domain which is independent of the particular choice of C up to Liouville isomorphism. If we assume in addition that h is complete, then $(U, -dd_c h)$ itself will be symplectically isomorphic to \hat{M} . In this context completeness of the gradient vector field can always be achieved by a reparametrization $h \rightarrow \beta(h)$ ([3], Lemma 3.1).

The notion of Liouville domain is closely related to that of *exact symplectic manifold with corners*, as defined in [16]. An exact symplectic manifold with corners $(M, \omega_M, \theta_M, I_M)$ is a compact smooth manifold with corners M , equipped with a symplectic form ω_M , a one-form θ_M satisfying $d\theta_M = \omega_M$, and an ω_M -compatible almost complex structure I_M . These should satisfy two convexity conditions: the

Liouville vector field must point strictly outwards along all boundary faces of M ; and the boundary must be weakly I_M -convex, which means that I_M -holomorphic curves cannot touch ∂_M unless they are completely contained in it.

Exact symplectic manifolds are technically more convenient when working with fibrations. Note that a Liouville domain M with a choice of compatible almost complex structure becomes an exact symplectic manifold (without corners) - all conditions except weak boundary convexity are automatic, and the maximum principle for holomorphic curves ensures that last condition as well. In the opposite direction, the only thing that will be important to us is that exact symplectic manifolds obtained in the course of our constructions will have at most codimension one corners and that such corners can be smoothed to make the resulting manifolds into honest Liouville domains (this is Lemma 7.6 in [16]; a similar smoothing occurs in the process of Weinstein handle attachment, [20]). All the invariants that we will consider will be insensitive to the details of these smoothings.

Chapter 3

Lefschetz fibrations.

Simply put, a Lefschetz fibration is a map with isolated singularities modeled on the complex singularity of the simplest type. The discussion below formalizes this description for the category of Liouville domains.

Most of the technical setup follows [16]. The sections most relevant for us are 15 and 16. We summarize what will be needed below.

3.1 Lefschetz fibrations.

We make the unit disc \mathbb{D}^2 into a Liouville domain by choosing the one-form $\eta = \frac{1}{2}r^2 d\theta$, so that $\eta_{\partial\mathbb{D}^2} > 0$ and $\omega_{st} = d\eta$. We give \mathbb{D}^2 the standard complex structure. This is the structure \mathbb{D}^2 inherits as a subdomain of \mathbb{C} . We will occasionally use Lefschetz fibrations over other subsets of \mathbb{C} , which also inherit Liouville structure from \mathbb{C} in the same way.

An (exact) Lefschetz fibration over \mathbb{D}^2 is a map from an exact symplectic manifold with corners $\pi : E \rightarrow \mathbb{D}^2$ which is I_E -holomorphic, with additional assumptions on behavior near boundary and the structure of critical points, as follows:

- Transversality to $\partial\mathbb{D}^2$.

At every point $x \in E$ such that $y = \pi(x) \in \partial\mathbb{D}^2$, we have $T\mathbb{D}^2 = T\partial\mathbb{D}^2 + D\pi(TE_x)$

This implies that $\pi^{-1}(\partial\mathbb{D}^2)$ is a boundary stratum of E of codimension 1, and we call it the vertical boundary of E , denoted by $\partial^v E$. The union of boundary faces of E not contained in $\partial^v E$ is the horizontal boundary of E , denoted $\partial^h E$.

- Regularity along $\partial^h E$.

If F is a boundary face of E not contained in $\partial^v E$, then $\pi|_F : F \rightarrow \mathbb{D}^2$ is a smooth fibration.

This implies that any fiber is smooth near its boundary.

- Horizontality of $\partial^h E$ with respect to the symplectic connection.

At any point x of E , we have $TE_x^v = \ker(D\pi_x)$. Away from critical points, the fact that π is I_E holomorphic implies that the symplectic complement TE_x^h of TE_x^v is transverse to it (and so defines a connection). We require that for all x in any boundary face F in $\partial^h E$ the horizontal TE_x^h is contained in TF_x .

- Lefschetz singularities.

We require that the critical points of π are generic (also called nondegenerate) and locally integrable. This means that I_E is integrable in a neighborhood of $\text{Crit}(E)$, and that $D\pi$ (seen as a section of the bundle $\text{Hom}_{\mathbb{C}}(TE; \pi^*T\mathbb{D}^2)$ of complex linear maps) is transverse to the zero- section. The second condition is equivalent to saying that the complex Hessian $D^2\pi$ at every critical point is nondegenerate. In addition, we will assume that there is at most one critical point in each fiber, so that the projection $\text{Crit}(\pi) \rightarrow \text{Crit}v(\pi)$ is bijective; this last assumption is for convenience only, and could easily be removed.

Nondegeneracy of critical points implies that they are isolated, so $\text{Crit}(\pi)$ is a finite subset of $\text{int}(E)$, and similarly $\text{Crit}v(\pi)$ a finite subset of $\text{int}(S)$. Locally near each critical point and its value, one has holomorphic coordinates in which π becomes the standard quadratic map $Q(x) = x_0^2 + \dots + x_n^2$. Generally ω_E will not be standard in these coordinates. However, one can find a deformation of the fibration which is well-behaved along $\partial_h E$ (and which in fact is local near the critical point), such that

at the other end of the deformation the Kähler form becomes the standard form in a given holomorphic Morse chart. This increases the importance of the following basic model (see also Figure 3-1):

Example 3.1.1. Let $Q : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be the quadratic $Q(x_0, \dots, x_n) = x_0^2 + \dots + x_n^2$, and $k : \mathbb{C}^{n+1} \rightarrow \mathbb{R}_{\geq 0}$ the function $k(x) = \frac{(|x|^4 - |Q(x)|^2)}{4}$. For some fixed $r, s > 0$ define $E = \{x \in \mathbb{C}^{n+1} : |Q(x)| \leq r; k(x) \leq s\}$, and equip it with the restriction of the standard symplectic form on \mathbb{C}^{n+1} , its standard primitive $\frac{i}{4}(zd\bar{z} - \bar{z}dz)$, the given complex structure $I_E = i$, and the map $\pi : E \rightarrow r\mathbb{D}^2$ obtained by restricting Q .

The boundary faces are $\partial_v E = \{x \in E : |\pi(x)| = r\}$, $\partial_h E = \{x \in E : k(x) = s\}$. The cutoff function k is chosen so as to make TE^h parallel to $\partial_h E$. To see that, one notes that TE_x^h is generated over \mathbb{C} by $(\nabla Q)_x = 2\bar{x}$, and checks that $dk_x(\bar{x}) = 0$, $dk_x(i\bar{x}) = 0$. Each nonsingular fiber E_z , $z \neq 0$, is symplectically isomorphic to the subset $B^*S \subset T^*S^n$ consisting of cotangent vectors of length (in the standard metric) at most \sqrt{s} . Explicitly, $B_s^*S^n = \{(u; v) \in R^{n+1} \times S^n : \langle u; v \rangle = 0, |u|^2 \leq s\}$, with the symplectic form $du \wedge dv$, and an isomorphism $E_z \rightarrow B_s^*S^n$ for $z > 0$ is given by $\phi_z(x) = (-\operatorname{Im}(x)|\operatorname{Re}(x)|; \operatorname{Re}(x)|\operatorname{Re}(x)|^{-1})$. This is discussed in more detail in the beginning of chapter 4. Unfortunately, while the negative Liouville (negative radial) vector field does point inwards along ∂E , it is not true that E is weakly I_E -convex (the fibers are, but not the total space). Hence, this is not quite an example of an exact Lefschetz fibration as defined here, even though from a purely symplectic viewpoint, it has all the desired features. Of course, one could change I_E to improve the situation, but there is no real point in doing that, since ultimately E will serve only as a local model.

The existence of this local normal form is a consequence of the holomorphic Morse Lemma (more precisely, the statement is that for any choice of holomorphic coordinates on the base, one can find coordinates on the total space in which $\pi = Q$). The deformation which allows one to make the symplectic structure standard in such coordinates is constructed in [11], Lemma 1.6.

3.2 Vanishing paths and cycles, Lefschetz thimbles.

For an (exact) Lefschetz fibration, we call an embedded curve $\gamma : [0, 1] \rightarrow \mathbb{D}^2$ a *vanishing path* if it avoids critical points except at the end, i.e. $\gamma^{-1}(\text{Crit} \pi) = 1$. To each such path we can associate its *Lefschetz thimble* which is the unique embedded Lagrangian $(n+1)$ -ball in E satisfying $\pi(\Delta_\gamma) = \gamma([0, 1])$. The boundary $V_\gamma = \partial\Delta_\gamma$, which is a Lagrangian sphere in $E_{\gamma(0)}$, is called the *vanishing cycle* of γ .

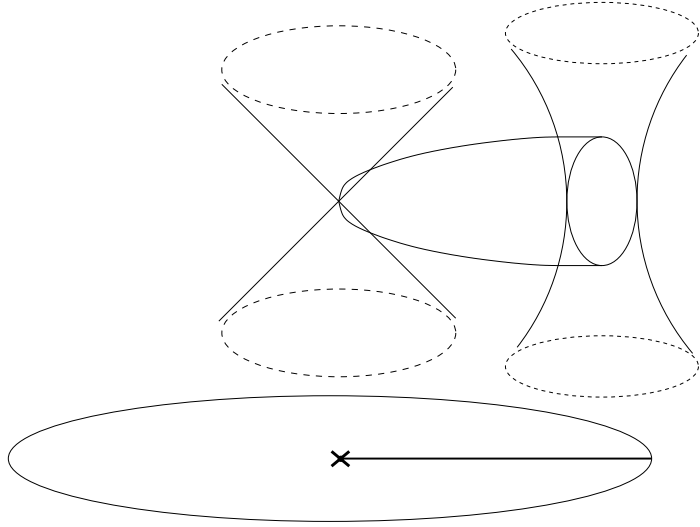


Figure 3-1: Model Lefschetz singularity.

Since it bounds a Lagrangian disc in E , any vanishing cycle is automatically exact. We refer to section 16 of [16] for the proof that such a thimble exists and is unique.

What is relevant for us is the Remark 16.4, which states that the vanishing cycle of any piece $\gamma|_{[t_0; 1]}$ is related to that of the whole by parallel transport:

$$V_\gamma = h_{\gamma|_{[0, t_0]}}^{-1}(V_{\gamma|_{[t_0, 1]}}).$$

Moreover, the vanishing cycle comes with an isotopy class of *framings* - diffeomorphism $v : S \rightarrow V$ of the standard sphere S (near the critical point this comes from the diffeomorphism with the sphere in tangent space at the critical point, and is then promoted by the parallel transport). This isotopy class is part of the data of the vanishing cycle.

We further have

Example 3.2.1. Take the model $\pi : E \rightarrow \mathbb{D}^2$ as defined in Example 3.1.1. This is not quite a Lefschetz fibration, but the missing condition (lack of holomorphic convexity)

is irrelevant for the present purpose. The only critical value is 0, and for any vanishing path γ , the Lefschetz thimble can be explicitly determined: $\Delta_\gamma = \bigcup_{0 \leq t \leq 1} \sqrt{\gamma(t)} S^n$.

Here $\sqrt{z} S^n = \{x \in C^{n+1} : x = \pm \sqrt{z} y \text{ for some } y \in S^n \subset R^{n+1}\}$. To see that this is the case, one uses the function k from Example 3.1.1, which is unchanged under parallel transport, and observes that $k^{-1}(0)$ is precisely the union of the subsets $\sqrt{z} S^n$ for all $z \in \mathbb{D}^2$. In the identifications of the fibers of the model fibration in Example 3.1.1 with sphere cotangent bundles, these are the zero sections. These spheres are what we will refer to as the “belt” spheres in what follows.

3.3 Matching cycles.

For (exact) Lefschetz fibration π consider an embedded path $\mu : [-1; 1] \rightarrow \text{int}(\mathbb{D}^2)$ such that $\mu^{-1}(\text{Critv}(\pi)) = \{-1; 1\}$. We can split this into a pair of vanishing paths with the same starting point, $\gamma_\pm(t) = \mu(\pm t)$ for $t \in [0; 1]$, hence get a pair of vanishing cycles $V_{\gamma_\pm} \subset M = E_{\mu(0)}$. When these two are equal (which is not going to be true on the nose in general, but suffices for the present applications) $\Sigma_\mu = \Delta_{\gamma_+} \cup \Delta_{\gamma_-}$ is a smooth Lagrangian submanifold of the total space E (by definition of the Lefschetz thimble, parallel transport along μ maps the intersections $\Sigma_\mu \cap \mu^{-1}(t)$ to each other for all $-1 < t < 1$, which gives a local chart $(-1; 1) \times V_{\gamma_\pm}$ around the overlap $\Delta_{\gamma_+} \cap \Delta_{\gamma_-} = V_{\gamma_\pm}$). Being the result of gluing two balls along their boundaries, Σ_μ is necessarily a homotopy sphere. In the case when the framings of the V_{γ_\pm} are isotopic, it is a standard sphere differentiably. In fact, given a choice of isotopy between the two framings, one can obtain a framing Σ_μ . We will refer to Σ_μ as the *matching cycle* (see also Figure 4-1).

We will apply this construction in the following context - given a Lefschetz fibration $\rho : F \rightarrow \mathbb{D}^2$, we can choose some matching paths μ_i , and, under fortunate circumstances, get framed Lagrangian spheres L_i in F . We then use the L_i ’s to construct another Lefschetz fibration $\pi : E \rightarrow \mathbb{D}^2$ with vanishing cycles L_i .

We note that in this case E is an instance of a *bifibration*. Bifibrations are discussed in some detail in section 15 of [16], but we will not use their theory in any systematic

way.

Finally we should note that the present discussion is somewhat simplified. Among other things, one can define Lefschetz fibration over any Riemann surface with boundary, and give a more robust definition of matching cycles. Both of these and more can be found in the main reference for section 3 - Seidel's book [16].

Chapter 4

The construction.

We will work with cotangent bundles of spheres T^*S^{n+1} . These have the standard embedding into $\mathbb{R}^{2(n+2)} = T^*\mathbb{R}^{n+2}$, via $T^*S^{n+1} = \{(q, p) \mid |q| = 1, q \cdot p = 0\}$, the derivative of the standard embedding of S^{n+1} into \mathbb{R}^{n+2} . In this model the standard symplectic form on T^*S^{n+1} is the restriction of the symplectic form $\omega = dp \wedge dq$ by naturality. However, for us a different model is going to be more convenient.

Namely, consider the conic $E = \{\sum z_j^2 = 1\}$ in \mathbb{C}^{n+2} . This is the fiber over 1 in the basic model of Lefschetz fibration (Example 3.1.1, see also [16], Example 15.9, or Lemma 1.10 in [11]). In terms of $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ in \mathbb{R}^{n+2} it is given by $|x|^2 - |y|^2 = 1$, $x \cdot y = 0$, and hence $(x, y) \mapsto (-\frac{x}{|x|}, y|x|)$ is a diffeomorphism to the sphere cotangent bundle.

We compute $dq_i = -\sum_j (\delta_{ij}|x|^{-1} + x_j(-x_i)|x|^{-3})dx_j$, so that $\sum_i p_i dq_i = -\sum_i y_i dx_i - \sum_{i,j} x_j y_j x_i |x|^{-2} dx_i = -\sum_i y_i dx_i$, where we used $\sum_j x_j y_j = 0$ in the last equality. The standard primitive of the symplectic form on \mathbb{C}^n is $\frac{i}{4}(z d\bar{z} - \bar{z} dz) = \frac{1}{2}(x dy - y dx)$. The difference between the pullback of pdq computed above and the restriction of this primitive is $x dy + y dx = d\langle x, y \rangle = 0$ on the conic. Hence we have an exact symplectomorphism of the conic and the standard cotangent bundle of the sphere. The inverse map is given by $(q, p) \mapsto (aq, \frac{1}{a}p)$, where $a^2 - \frac{1}{a^2}|p|^2 = 1$, so $|p|^2 = a^4 - a^2$, $a \geq 1$.

We want to use the conic model from now on, but technically it does not fit with our definitions - we want to work with a compact exact symplectic manifold with

boundary, but the conic (as well as the cotangent bundle itself) is non-compact. This is remedied by taking a bounded part where $|p|^2 < s$. Note that $|p|^2$ is precisely $k(x)$ in Example 3.1.1. Since we are on the fiber $Q(x) = 1$, via the symplectomorphism above this corresponds to $|z| < (1 + 4s)^{\frac{1}{4}}$. We denote $(1 + 4s)^{\frac{1}{4}}$ by r , and assume that s is large, and hence so is r . Then the resulting manifold has an inward pointing Liouville flow and is weakly convex with respect to the standard complex structure.

We now build a Lefschetz fibration for this conic model. Consider the projection to the last coordinate $\pi : E \rightarrow \mathbb{C}$ sending (z_1, \dots, z_{n+2}) to z_{n+2} . Perhaps the easiest way to understand it is to note that the fiber over λ is given by $z_1^2 + \dots + z_{n+1}^2 = 1 - \lambda^2$, and so it is in fact a pull back via $\lambda \rightarrow 1 - \lambda^2$ of the canonical Lefschetz local model fibration of Example 3.1.1 in one dimension lower.

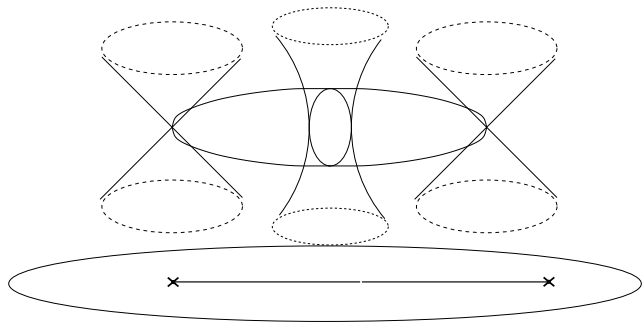


Figure 4-1: Lefschetz fibration for T^*S^n .

In particular it has critical points when $z_1 = \dots = z_{n+1} = 0$ and $1 - z_{n+2}^2 = 0$ i.e. $z_{n+2} = 1$ or $z_{n+2} = -1$; these critical points are nondegenerate; the corresponding critical fibers are conical; and the smooth fibers are exact symplectomorphic to the cotangent bundles of the sphere in one dimension lower, T^*S^n . Moreover if we take a reference fiber above $\lambda = 0$ and straight line vanishing paths (the intervals $[-1, 0]$ and $[0, 1]$), these paths both come from the straight line vanishing path from 0 to 1 in the model fibration, and so the corresponding thimbles are made up of “belt” spheres in T^*S^n ’s and the two vanishing cycles agree and are equal to the belt sphere of the fiber over 0. The union of the corresponding two thimbles is the zero-section S^{n+1} of the total space. We note that in fact any reference fiber and any vanishing path produce the relevant belt sphere as vanishing cycles in this setup (see Figure 4-1).

We now restrict to $|z| < r$. The portion of the fiber over z_{n+2} with $|z| < r$ is symplectomorphic to the disc cotangent bundle of a sphere of radius squared $w =$

$\frac{1}{4}((r^2 - |z_{n+2}|^2)^2 - |1 - z_{n+2}^2|^2) = \frac{1}{2}(\frac{r^4-1}{2} - x_{n+2}^2(r^2-1) - y_{n+2}^2(r^2+1))$. In particular, the piece of T^*S^{n+1} with $|z| < r$ projects to the inside of the ellipse $w \geq 0$ with foci at $z_n = 1$ and $z_n = -1$ and axes of length $\sqrt{\frac{r^2+1}{2}}$ and $\sqrt{\frac{r^2-1}{2}}$. We take a smaller ellipse $w \geq \epsilon^2$ and restrict our fibration to it and to the subset of the total space which corresponds to disc bundles of radius ϵ over the corresponding fibers. We think of ϵ as being small compared to s and r but big compared to 1.

The Liouville flow of T^*S^{n+1} is everywhere outward pointing along the boundary of this subset, and after smoothing the corner as in Lemma 7.6 in [16] and completing, we recover the Liouville domain T^*S^{n+1} .

We have therefore constructed an exact Lefschetz fibration with total space a piece of the disc bundle of T^*S^{n+1} . We now look at our reference fiber and realize it as the total space of another, auxiliary, fibration.

In fact, the reference fiber $z_{n+2} = 0$ is an n -sphere cotangent bundle, and is simply our model conic in dimension n . As such it admits a Lefschetz fibration by projection to z_{n+1} with two critical points corresponding to $z_{n+1} = 1$ and $z_{n+1} = -1$ and exactly the same structure as described above, only in one dimension lower.

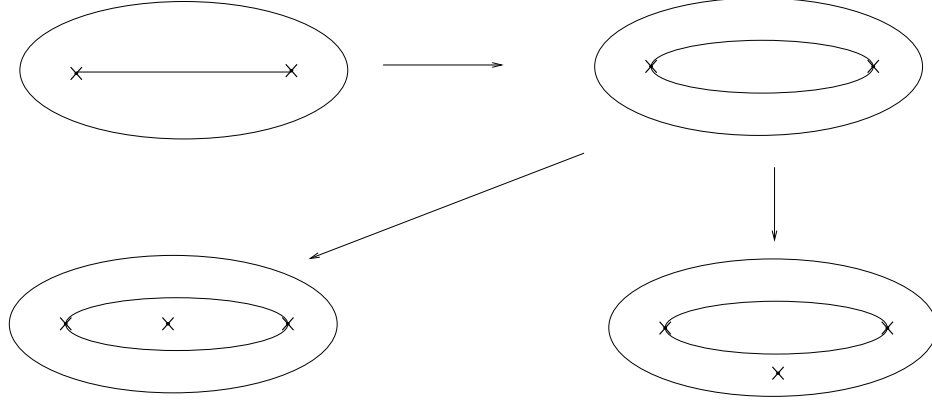
To recap, all of this gives the following description of T^*S^{n+1} - it is the total space of a Lefschetz fibration π over a disc with critical values $+1$ and -1 with a fiber T^*S^n and vanishing cycles equal to the zero section S^n . We call π the main fibration. The reference fiber itself is also the total space of a Lefschetz fibration ρ with two critical values $+1$ and -1 and the vanishing cycles of the main fibration π are matching cycles for the straight matching path between the critical values of ρ . We call ρ the auxiliary fibration.

We will construct two Liouville domains W_0 and W_1 by starting with this description of T^*S^{n+1} and modifying it in stages. See Figure 4-2. At each stage we modify either the vanishing cycles of the main fibration or its fiber. Lemma 16.9 in [16] then tells us that we can build the the corresponding total spaces. Both W_0 and W_1 will be obtained in this manner.

The first step is to change the matching paths for the auxiliary fibration ρ . The exact choice of paths is immaterial, for definiteness we can take the upper and lower

U is the sphere cotangent, both vanishing cycles are the zero section.

U is still the sphere cotangent, vanishing cycles Hamiltonian isotopic.



U has a handle attached, vanishing cycles not Hamiltonian isotopic.

U has a handle attached, vanishing cycles Hamiltonian isotopic.

Figure 4-2: Building W_0 and W_1 .

semicircular arcs of $|z_{n+1}| = 1$, which we will denote by α and β . This produces matching cycles A and B in T^*S^n that are framed Lagrangian isotopic to the ones we had before (the zero section), and the uniqueness counterpart of Lemma 16.9 in [16] implies that the total space of the main fibration, obtained from the same fiber as before and framed Lagrangian isotopic vanishing cycles, after corner smoothing and completion, is Liouville isomorphic to T^*S^{n+1} .

The second stage is more substantial. We change the fiber of the main fibration, i.e. the total space of the auxiliary fibration. The auxiliary fibration has two critical points 1 and -1 , and for straight vanishing paths to a reference fiber at, say $-3i$ the vanishing cycles are the zero-sections of the fiber T^*S^{n-1} over $-3i$. We modify this by adding a third critical value at $-2i$ for W_0 and at 0 for W_1 , such that the vanishing cycle for a straight line vanishing path is equal to the belt sphere (in both cases). Note that in the construction for W_1 the new critical value lies inside the disc encircled by the matching paths that define the Lagrangian spheres A and B , while and in the construction for W_0 it lies outside. By successive applications of Lemma 16.9 in [16], we get corresponding total spaces for the auxiliary fibrations and then for the main fibrations. Note that the total spaces of the auxiliary fibrations U_0 and U_1 are exact symplectomorphic, the only difference is in the matching cycles that specify

how the total space of the main fibration is built.

We now show that W_0 and W_1 are the same as smooth manifolds. In section 6, we will show that W_0 and W_1 are not exact symplectomorphic.

Proposition 4.0.1. *The smooth manifolds W_0 and W_1 are diffeomorphic.*

Proof. Let's consider the construction of W_i 's in more detail. Lemma 16.9 in [16] describes the process of building the total space of the Lefschetz fibration as a sequence of surgeries - one first thickens the fiber by taking its product with \mathbb{D}^2 and then performs a series of handle attachments along the spheres given by the vanishing cycles (see also [10]). It is therefore sufficient to show that the matching cycles A and B are smoothly isotopic.

The matching cycle A is the union of belt spheres of fibers over the matching path α and the matching cycle B is the union of belt spheres of fibers over the matching path β (plus the two critical points).

The paths α and β are isotopic in \mathbb{D}^2 (relative the endpoints). For $\alpha, \beta : [-1, 1] \rightarrow \mathbb{D}^2$ we have the isotopy $\gamma : [-1, 1] \times [0, 1] \rightarrow \mathbb{D}^2$. We can take γ to be symmetric, that is to satisfy $Re(\gamma(t, \cdot)) = -Re(\gamma(-t, \cdot))$, and to stay inside the disc encircled by α and β . Again, the exact choice of γ is immaterial, but for definiteness we can take each $\gamma(\cdot, s)$ to be a uniformly parametrized circular arc through the two critical points and $i(1 - 2s)$. For each time $s \in [0, 1]$ if the path $\gamma(\cdot, s)$ misses the third critical point at the origin, it defines a matching cycle Γ_s . The only problem occurs when γ_s hits the origin and the belt sphere over $\gamma(0, s)$ shrinks to a point (for our choice of γ this happens at $s = \frac{1}{2}$). To remedy this, we push the Γ_s off the zero section, see Figure4-3.

The details are as follows. It is well-known that for even n the sphere S^{n-1} has a smooth vector field δ with $|\delta| = 1$ on it (in the standard metric). Take a smooth "horizontal" cutoff function $h(t) : [-1, 1] \rightarrow [0, \frac{1}{2}\epsilon]$, which is zero near the endpoints and equal to $\frac{1}{2}\epsilon$ near 0; take also a smooth "vertical" bump function $v(s) : [0, 1] \rightarrow [0, 1]$.

Consider B the disc enclosed by α and β with small (contractible) neighborhoods of the critical points taken out. Before we added the third critical point at zero, the

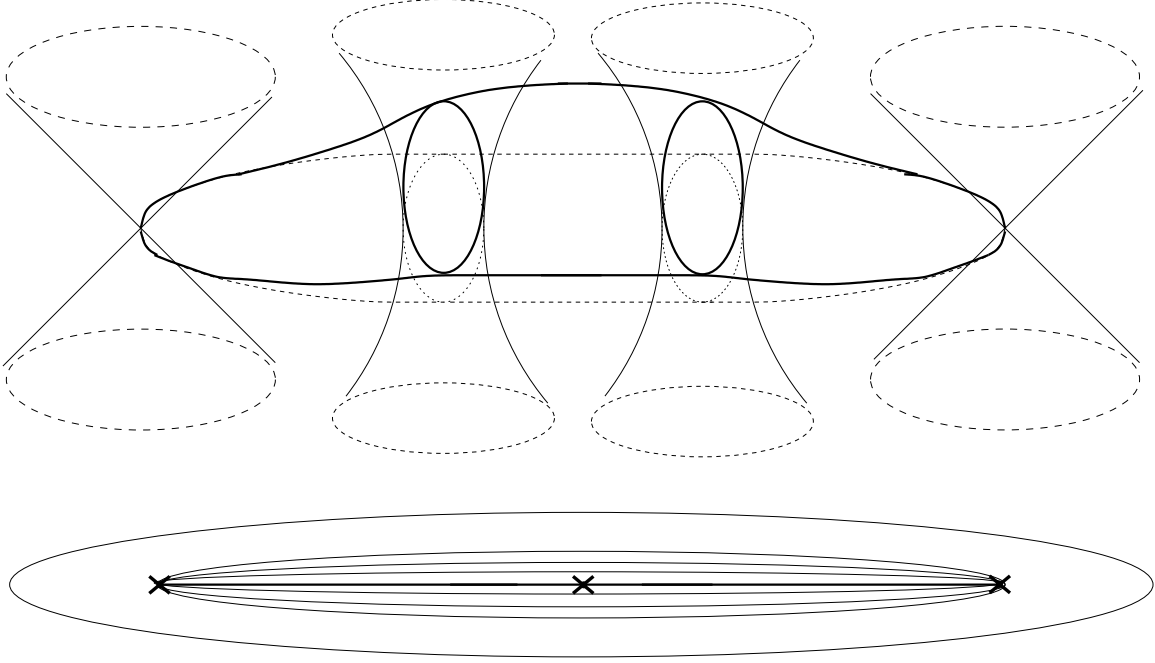


Figure 4-3: Pushing the isotopy.

fibration ρ was (smoothly) trivial over B , with a trivialization

$$\Psi : B \times \{(v \in T^*S^{n-1} ||v| < \epsilon)\} \rightarrow \rho^{-1}(B)$$

taking belt spheres to belt spheres. Define for all t such that γ_s is in B , the set $L(t, s) = \{h(t)v(s)\delta(p) | p \in S^{n-1}\}$. This is a push-off of the zero section over $\gamma(t, s)$ in the direction of δ .

We use the trivialization Ψ to define $\hat{\Gamma}_s$ as the union of $L(t, s)$ over all such t in the fiber over $\gamma(s, t)$ together with the belt spheres of ρ over the parts of γ_s lying outside B (these glue smoothly because $h(t)$ is zero near the endpoints). This is a sphere in the total space of ρ - the push off of the whole matching cycle of γ_s in the direction of δ . Now the union of $\hat{\Gamma}_s$ over all s gives an isotopy of the matching cycles A and B . Adding the third critical point at zero happens as a surgery on the belt sphere at zero, supported in its neighborhood. The spheres $\hat{\Gamma}_s$ stay away from the surgery region and hence persist in the manifold U_1 , defining isotopy between the matching cycles A and B in it.

This means that the total space W_1 is the same smooth manifold as the space

obtained by attaching handles to thickened U_1 along two copies of the matching cycle A . But that is the same as the total space W_0 . This completes the proof.

□

Remark 4.0.2. In the case of W_1 , if we denote by L the matching cycle of the straight matching path from -1 to 0 and by R the matching cycle of the path from 0 to 1 , we see by Lemma 16.13 in [16] that the bottom matching cycle B is obtained by the Dehn twist around R of L , and the top matching cycle A is obtained by the inverse Dehn twist around R of L . So B is obtained from A by the square of the Dehn twist around R . For spheres of dimension 2 and 6, the square of the model Dehn twist is smoothly isotopic to identity (which of course implies that A is isotopic to B). The case $n = 2$ is Lemma 6.3 in [17], and the case $n = 6$ can be handled somewhat analogously by using the almost-complex structure on S^6 . The case of other n appears to be open.

Chapter 5

Wrapped Floer homology and wrapped Fukaya category.

Wrapped Floer cohomology is an adaptation of the usual Lagrangian Floer homology to the context of non-compact Lagrangians in Liouville domains. The main concern in this situation is what to do about intersections at infinity. We restrict to Lagrangians with exact cylindrical ends, which we will call admissible. In this setting, for L_1 and L_2 admissible, we have two possible solutions. We can perturb one of them slightly via Reeb flow near the boundary. We will denote this by $HF(L_1, L_2)$. Alternatively, we can “wrap it around” by long-time Reeb flow, observing that all resulting complexes form a direct system (with respect to the time parameter), and taking the direct limit. The resulting homology is invariant under isotopies of L_i ’s among admissible Lagrangians, it is called the wrapped Floer cohomology and is denoted by $HW(L_1, L_2)$. In the case where the ambient manifold is a Lefschetz fibration over the disc and L_i are Lefschetz thimbles, the result of the flow is literally wrapping the vanishing path around the disc. See Figures 5-1 and 5-2.

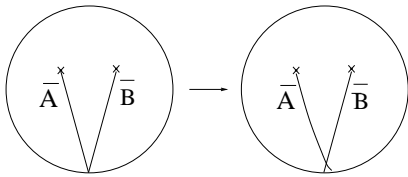


Figure 5-1: $HF(\overline{A}, \overline{B})$

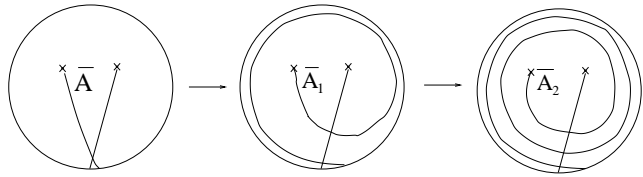


Figure 5-2: $HW(\overline{A}, \overline{B}) = \lim HF_m(\overline{A}, \overline{B})$

Wrapped Floer cohomology can be viewed as a Lagrangian (or open-string) version of symplectic cohomology introduced by Viterbo ([18],[19]). It uses the same class of Hamiltonians with linear slope at infinity and the same direct limit procedure.

Let's discuss this construction in the context suitable for Lefschetz fibrations in some more detail. Our setup differs in some technical details from the ones in the literature, for example in [1] section 3, but is similar. We repeat some of the definitions here with modifications suitable for the context of Lefschetz fibrations.

Firstly, we need to decide on a class of admissible Lagrangians. There are several options (see [1]), but in our application the Lagrangians are going to be Lefschetz thimbles, which would be admissible in any one of them.

For the sake of definiteness, we will call a Lagrangian L admissible if it intersects ∂M transversally and is exact, meaning $\theta|_L$ is exact, $\theta|_L = df$. Moreover we require that $\theta|_L$ vanishes on a neighborhood of ∂L . Near its boundary, this makes L a cone over the Legendrian submanifold ∂L of ∂M and allows us to attach an infinite cone, extending L to \hat{L} . For thimbles this corresponds simply to the vanishing path going radially in a straight line near the boundary of \mathbb{D}^2 , thus allowing to extend it by a straight ray.

Secondly, we need to choose an appropriate family of Hamiltonians. Recall that for a Liouville domain M , the Liouville flow κ gives the collar neighborhood $\kappa : (-\infty, 0] \times \partial M \rightarrow M$ with $\kappa^*\theta = e^s\alpha$ and $\omega = d\theta$, and this is extended to positive s in the completion. Abouzaid and Seidel use the coordinate $r = e^s$ on the cone and a class of Hamiltonians $H \in C^\infty(M, \mathbb{R})$ which are everywhere positive and admit a smooth positive extension \hat{H} to \hat{M} such that $\hat{H}(r, y) = r$ on the semi-infinite cone. Then if \hat{X} is the Hamiltonian vector field of \hat{H} , then on the cone $\hat{X} = (0, R)$ where R is the Reeb vector field of the contact one form $\theta|_{\partial M}$.

This choice of coordinates for the infinite cone has the following slightly unfortunate consequence. Viewing \mathbb{C} as the completion of \mathbb{D}^2 with its standard Liouville structure, we get $\omega = r_{polar} dr_{polar} \wedge d\theta = d(\frac{1}{2}r_{polar}^2 d\theta)$ so that $r = \frac{1}{2}r_{polar}^2$. In particular a Hamiltonian linear in r is quadratic in r_{polar} . In discussing Hamiltonians on \mathbb{D}^2 and \mathbb{C} we will work with the completion coordinate r , writing r_{polar} whenever we refer to

the standard polar radius.

We will consider the following family of Hamiltonians. Consider a Hamiltonian H_b on the base disc of the fibration which is radial, zero in the interior of the disc and of slope one in r near the boundary. For a Lefschetz fibration over unit disc \mathbb{D}^2 , we have $H_b(re^{i\theta}) = \eta(r)$, for $\eta(r) : [0, 1] \rightarrow [0, \infty)$ with $\eta(r) = 0$ for $r < 1 - 2\epsilon$, and $\eta'(r) = 1$ for $r > 1 - \epsilon$. Further, define H to be the pull back of H_b under the fibration map. Then H_b extends to all of \mathbb{C} linearly in r , and H extends linearly to \hat{H} on the part of the infinite cone attached to the vertical boundary of the fibration. This Hamiltonian is not of the right type near the horizontal boundary and the corners. This is of no importance. All our Lagrangians and all holomorphic curves in all moduli spaces that we consider stay away from the horizontal boundary and corners. Hence we can modify our Hamiltonian near those boundaries to make our construction fall in line with the general theory. We assume that this has been done, so that all Hamiltonians that appear henceforth are linear on all of the attached infinite cone. Alternatively, we could simply work directly with our original Hamiltonians, without any difference in results; the only concern with that approach is psychological difficulties in relating the resulting theory to the standard wrapped Floer cohomology constructions. After all we want to view the thimbles as elements of the usual wrapped Fukaya category, which requires the use of Hamiltonians linear at infinity.

The Hamiltonian H_b generates a flow which is supported near the boundary of \mathbb{D}^2 and acts by rotations. We have $H = \pi^*H_b$, $dH = \pi^*dH_b$, $dH(v) = \pi^*dH_b(v) = dH_b(\pi_*v)$, so that $\omega(X_H, v) = \omega_b(X_{H_b}, \pi_*(v))$. This means that X_H is in the symplectic complement of the fiber of π (take vertical v), and π_*X_H is parallel to X_{H_b} (take $v = X_H$). The proportionality constant is given by the norm of the derivative of π and can be made the same on all points of each belt sphere of any fiber. Correspondingly, the flow of \hat{H} is the rescaled horizontal lift of the flow of H_b (with respect to the symplectic connection) and is given by parallel transport so that the image of a Lefschetz thimble under the flow of \hat{H} is again a Lefschetz thimble, over a path that has been flowed under a rescaling of H_b . Up to scaling by a constant factor we can assume that the time 1 flow of \hat{H} wraps the thimbles exactly once around $\partial\mathbb{D}^2$ at the

boundary.

We now consider a family of Hamiltonians $\hat{H}_m = (m + \delta)\hat{H}$, for positive integers m and a fixed positive small real δ , chosen in such a way that the images under the flow of \hat{H}_m of relevant thimbles are again thimbles but over paths that don't intersect on the boundary of \mathbb{D}^2 . Denote the flow of \hat{H}_m by ψ_m . Then all the intersections of the thimbles lie in the fibers over the intersection points of their paths in the interior of the disc, and are given by intersections of the corresponding vanishing cycles (for the case of the thimble intersecting with itself, this may require an additional compactly supported isotopy of the vanishing path to make it transverse to itself, which corresponds to a Hamiltonian isotopy of the thimble and does not affect the Floer cohomology). After a small perturbation we can assume that the relevant vanishing cycles intersect each other transversely. Then for a pair of thimbles \overline{A} and \overline{B} we have for each m the usual Floer-theoretic finitely generated chain complex $CF_m(\overline{A}, \overline{B}) := CF(\psi_m(\overline{A}), \overline{B})$, with a differential obtained by counting pseudoholomorphic strips, which computes the Floer cohomology $HF_m(\overline{A}, \overline{B}) := HF(\psi_m(\overline{A}), \overline{B})$ in the usual way. Again as usual, this comes with a product structure $HF_{m_1}(\overline{A}, \overline{B}) \otimes HF_{m_2}(\overline{B}, \overline{C}) \rightarrow HF_{m_1+m_2}(\overline{A}, \overline{C})$ obtained by counting triangles (there is a slight problem, as the composition of the two Hamiltonian isotopies on the left is not the Hamiltonian isotopy on the right on the nose, but the difference is a small Hamiltonian isotopy, which induces isomorphism on the Floer cohomology. See also the discussion in section 1.1 of [8]). Moreover, as in the case of symplectic cohomology, positivity of H ensures the existence of continuation chain maps $\kappa_n^m : CF_m(\overline{A}, \overline{B}) \rightarrow CF_n(\overline{A}, \overline{B})$ for all $n > m$, so that $CF_m(\overline{A}, \overline{B})$ and $HF_m(\overline{A}, \overline{B})$ form a direct system (see [15] for the case of symplectic cohomology). The convexity conditions on \overline{A} , \overline{B} and M ensure that the relevant pseudoholomorphic curves stay away from the boundary and so the appropriate moduli spaces are compact. The rest of the analysis is the same as in the case of compact Lagrangians. We note that transversality can be achieved by perturbing the almost complex structure J on M “in the vertical direction”, that is keeping the projection map of the Lefschetz fibration pseudoholomorphic. Since direct limits are exact, we can define the wrapped Floer cohomology of \overline{A} and \overline{B} as either the cohomology of the direct

limit of the chain complexes or simply as the direct limit of the cohomology system, $HW(\overline{A}, \overline{B}) = \varinjlim HF_m(\overline{A}, \overline{B})$.

We note that in our circumstances of contractible thimbles the Stiefel-Whitney classes of the Lagrangians are manifestly zero, and the relative Chern class lives in $H_2(W)$, which is zero if $\dim W > 6$, and so in that case Floer cohomology is defined over \mathbb{Z} and \mathbb{Z} -graded, although since our proof is based only on computing the ranks of various Floer cohomology groups, this is of marginal importance. In fact since we use the Seidel long exact sequence from [11], we should work with $\mathbb{Z}/2$ coefficients and ungraded cohomology groups.

We should also mention that in the work of Abouzaid and Seidel [1] a slightly different version of wrapped Floer homology is given. There the authors are concerned with the A^∞ structure on the chain complex, and need a finer model than the direct limit construction. We, on the other hand, are only concerned with the cohomology and product structure, so a cruder, simpler model is sufficient. The resulting cohomology is the same in both models (see Lemma 3.12 in [1]).

Chapter 6

Distinguishing W_0 and W_1 .

Recall that in section 4 we have constructed, for all even n , two Lefschetz bifibrations with total spaces W_0 and W_1 which are both diffeomorphic to T^*S^{n+1} with a single subcritical handle added. In this section we prove that the wrapped Floer cohomologies of the Lefschetz thimbles are non-zero for W_0 and zero for W_1 . This proves Theorem 1.0.1.

We will use exact triangles to compute ranks of various Floer cohomology groups. We use the following:

Observation 6.0.3. If

$$\rightarrow K \xrightarrow{F} L \rightarrow M \rightarrow$$

is an exact triangle with $\text{rank } K = k$, $\text{rank } L = l$ and $\text{rank } \text{Im } F = f$, then $\text{rank } M = k + l - 2f$.

Call the vanishing cycles of the main fibration A and B . These are Lagrangian spheres in the fiber. The main tool we use is a comparison of Seidel's exact triangles in the fiber and in the total space. In the fiber we have for any exact Lagrangian submanifolds L_1 and L_2 and any framed lagrangian sphere L ([11], Theorem 1)

$$\rightarrow HF(L, L_2) \otimes HF(L_1, L) \rightarrow HF(L_1, L_2) \rightarrow HF(L_1, \tau_L L_2) \rightarrow$$

When $L_1 = L_2 = B, L = A$ we get

$$\rightarrow HF(A, B) \otimes HF(B, A) \xrightarrow{m} HF(B, B) \xrightarrow{a} HF(B, \tau_A B) \rightarrow$$

Here, m is the pair-of-pants product. Note that in our situation $HF(B, B) = H(S^n)$ has rank 2 by the classical computation of Floer. In the case of W_0 when A and B are Hamiltonian isotopic, Floer product reduces to cup product on cohomology and is onto. The map a is then zero by exactness, and the group $HF(B, \tau_A(B))$ is of rank 2.

Note that the spheres A and B intersect transversely at the two critical points of ρ which have the same grading mod 2. Hence Floer differential vanishes, and rank $HF(A, B)$ is equal to 2 even for the case of W_1 .

On the other hand for the matching paths in W_1 , the vanishing cycles are not Hamiltonian isotopic. In fact, they are not isomorphic as objects of the Donaldson-Fukaya category of the fiber. To see this consider the Lefschetz thimble L for the critical point over 0 of the auxiliary fibration ρ and the vanishing path going straight down. We see that in the total space of ρ , i.e the fiber of π , A and L are disjoint and so $HF(A, L) = 0$. On the other hand, since the vanishing path for L intersects the matching path for B exactly once, $HF(B, L)$ is the same as the Floer cohomology of the corresponding vanishing cycles (of ρ), that is of the vanishing sphere with itself. This is again $H(S^{n-1})$, and so is not zero.

Correspondingly, for these non-isomorphic vanishing cycles the pair of pants product still hits the fundamental class in $HF(B, B)$ (by Poincare duality in Floer theory). We want to see that the product misses the identity, which in turn forces the group $HF(B, \tau_A(B))$ to be of rank 4.

To see that m misses the identity, suppose there is $\alpha \in HF(A, B) \otimes HF(B, A)$ such that $m(\alpha) = Id \in HF(B, B)$. Then for a non-zero element $\beta \in HF(B, L)$, the composition of Floer products going from $HF(B, L) \otimes HF(A, B) \otimes HF(B, A)$ to $HF(B, L)$ on the one hand takes $\beta \otimes \alpha$ to $m(\beta, Id) = \beta$, and on the other hand factors through the group $HF(A, L) = 0$. As $\beta \neq 0$, this is a contradiction.

Now in the total space, denoting the thimbles by \overline{A} and \overline{B} and viewing them as objects of the derived Fukaya-Seidel category ([16]) of the Lefschetz fibration $W_i \rightarrow D^2$, the results of [13] imply that the result of twisting \overline{B} by monodromy, which we shall denote by \overline{B}_1 , is isomorphic to the cone of the evaluation map $ev : Hom(\overline{A}, \overline{B}) \otimes \overline{A} \mapsto \overline{B}$. Taking the corresponding exact triangle and taking the long exact sequence corresponding to the functor $Hom(\cdot, \overline{B})$ to it we get

$$\rightarrow Hom_{FS}(\overline{B}_1, \overline{B}) \rightarrow Hom_{FS}(\overline{B}, \overline{B}) \xrightarrow{ev^*} Hom_{FS}(\overline{A}, \overline{B})^* \otimes Hom_{FS}(\overline{A}, \overline{B}) \rightarrow$$

Here $Hom_{FS}(\overline{B}, \overline{B})$ has rank one and is generated by the identity. This is a general fact about Lefschetz thimbles - a vanishing path can be isotoped to intersect itself only at the critical point, which changes the thimble by a Hamiltonian isotopy ([2], Lemma 3.2), so the resulting cohomology group has rank 1.

The group $Hom_{FS}(\overline{A}, \overline{B})$ has the same generators as $HF(A, B)$. Since the Lefschetz fibration map π is pseudoholomorphic, any pseudoholomorphic strip in the total space has to project to a holomorphic strip on the base \mathbb{D}^2 with boundary on corresponding vanishing paths, and the maximum principle for \mathbb{D}^2 implies that any such disc is contained in the fiber. So not only generators, but also the differentials agree, and rank of $Hom_{FS}(\overline{A}, \overline{B})$ is 2.

The map ev^* is non-zero and maps $Id \mapsto \sum_{\alpha \in Hom_{FS}(\overline{A}, \overline{B})} \alpha \otimes \alpha^*$. This means the rank of $Hom_{FS}(\overline{B}_1, \overline{B})$ is 3.

The chain complex computing $Hom_{FS}(\overline{B}_1, \overline{B})$ contains the generators of the complex computing the group $HF(B, \tau_A B)$ and one additional generator u corresponding to the critical point of the main fibration, where the thimbles \overline{B} and \overline{B}_1 meet. Again, since π is pseudoholomorphic, any pseudoholomorphic strip in the total space has to project to a holomorphic strip on the base \mathbb{D}^2 with boundary on corresponding vanishing paths. The maximum principle for \mathbb{D}^2 implies that there are no strips connecting u to other generators (note that we are doing Floer cohomology, so for counterclockwise wrapping the differential from less wrapped thimble to the more wrapped one goes towards the critical point), and that any map not ending in u projects to a constant

map, i.e. is contained in the fiber. So all the generators other than u indeed form a quotient complex computing $HF(B, \tau_A B)$. The only question is whether u is in the image of the differential.

Comparing the ranks of the homology groups we see that for standard T^*S^{2n+1} and for W_0 the generator u must not be in the image of the differential, and so u survives in cohomology, whereas for the non-standard modification W_1 the generator u is in the image. Since the continuation maps commute with the differential, image of u under continuation maps stays in the image of the differential for the more wrapped $CW(\overline{B}_n, \overline{B})$, and so vanishes in cohomology $HW(\overline{B}, \overline{B})$.

However u represents the unit in $Hom_{FS}(\overline{B}, \overline{B})$, and as the continuation maps are compatible with the triangle products (see Section 3.3 of [8]), the image of u in $HW(\overline{B}, \overline{B})$ is the unit there. Hence the unit is zero, which is only possible if $HW(\overline{B}, \overline{B}) = 0$. A similar argument implies that $HW(\overline{A}, \overline{A}) = 0$.

Since $HW(\overline{A}, \overline{B})$ is a module over $HW(\overline{A}, \overline{A})$ it also vanishes.

This proves our main Theorem 1.0.1.

We note that this behavior of wrapped Floer cohomology is in sharp contrast to the one in standard cotangent bundles. There for F_i the cotangent fiber at the point p_i the wrapped Floer homology $HW(F_1, F_2)$ is the homology of the path space from p_1 to p_2 (see Theorem 3.2 in [8]). In fact by analogy with the result of Cieliebak that states that subcritical handle attachment does not change the symplectic cohomology ([4]), we expect that in the case of subcritical handle attachment the functor constructed by Abouzaid and Seidel in [1] is a full embedding, in which case the wrapped Floer cohomologies in W_0 should coincide with those in T^*S^{n+1} from which it is obtained. Meanwhile we have the corollary 1.0.2, which states that the manifold W_1 does not contain any closed exact Lagrangian submanifold. To see this, observe that, for such a Lagrangian L we would have on one hand, by Floer's original result (which still holds in the context of exact Lagrangians), $HF(L, L) = H^*(L)$, which is nonzero. On the other hand, $HF(L, L) = HW(L, L)$ since wrapping does not affect closed Lagrangian submanifolds, but by Theorem 4 of [7] there is a spectral sequence converging to $HF(L, L)$ with the first page $E_1^{jk} = (HF(\Delta_j^!, L) \otimes HF(L, \Delta_j))^{j+k}$ for a basis of

thimbles Δ and dual thimbles $\Delta^!$. However, $HF(L, \Delta_j) = HW(L, \Delta_j)$, again because L is closed, and the later group vanishes since it is a module over $HW(\Delta_j, \Delta_j) = 0$. Hence, the above results imply that this first page vanishes, a contradiction.

Since W_0 contains the exact Lagrangian sphere inherited from the zero-section of T^*S^{n+1} , we see that W_0 and W_1 are not exact deformation equivalent.

We also note that were the wrapped Floer homology $HW(\overline{A}, \overline{A})$ to vanish in the case of W_0 , then by symmetry so would $HW(\overline{B}, \overline{B})$, and we could repeat the above argument. So the fact that there is an exact Lagrangian sphere in W_0 implies that these groups are non-zero.

Chapter 7

Possible extensions.

The work in this thesis is only the beginning of an investigation of the symplectic invariants of Lefschetz fibrations. The most immediate extension is to create “fakes” of other manifolds. In fact, the process of stabilization allows us to embed a configuration of thimbles similar to the one we worked with above into any Lefschetz fibration. An investigation of these situations can potentially lead to a construction of “symplectic fakes” of the total space of any $4n + 2$ dimensional Lefschetz fibration. Furthermore, it should be possible to avoid adding any subcritical handles, getting manifolds diffeomorphic to the original ones on the nose.

Conjecture. *For any Liouville domain W of dimension $4n+2$ there exists a Liouville domain W' such that*

- *W and W' are diffeomorphic*
- *The symplectic homology and wrapped Fukaya category of W' are zero.*

Another direction is to investigate related Floer-theoretic invariants of these Liouville domains. In particular, a conjecture of Paul Seidel relates the wrapped Fukaya category of a Lefschetz fibration to the category of modules over a certain curved A_∞ - algebra \mathcal{D} defined in terms of the vanishing cycles [14], and another conjecture states that the symplectic homology of the total space is the Hochschild homology

of \mathcal{D} . Together these conjectures mean that vanishing of the wrapped Floer cohomologies for all thimbles implies vanishing of the symplectic cohomology of the total space. While these conjectures are unproved at the moment, a long exact sequence describing the behavior of symplectic homology under critical handle attachment has been recently proved by Bourgeois, Ekholm and Eliashberg and should give a good approach to proving the vanishing of $SH(W_1)$ directly.

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